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2000 J. Phys. A: Math. Gen. 33 4129

(http://iopscience.iop.org/0305-4470/33/22/315)

J. Phys. A: Math. Gen. **33** (2000) 4129–4141. Printed in the UK PII: S0305-4470(00)12629-0

Chaotic Aharonov–Bohm scattering on surfaces of constant negative curvature

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Received 9 March 1999

Abstract. A topological model of the Aharonov–Bohm scattering is presented, where the usual set-up is modelled by a genus-one Riemann surface with two cusps, i.e. leaks infinitely far away. This constant negative-curvature surface is uniformized by the Hecke congruence subgroup $\Gamma_0(11)$ of the modular group. The fluxes through the holes are described by the even Dirichlet character for $\Gamma_0(11)$. The scattering matrix having only off-diagonal elements (no reflection) is calculated. The fluctuating part of the off-diagonal entries shows a non-trivial dependence on the fluxes as well. The scattering resonances are related to the non-trivial zeros of a Dirichlet L-function. The chaotic nature of the scattering is related to the distribution of primes in arithmetical progressions.

1. Introduction

Two-dimensional multiply connected surfaces of constant negative curvature are widely used in physics. They are studied by string theorists for calculating multiloop amplitudes in string perturbation theory [1]. The other topic of interest is quantum chaos. It is well known by now that geodesic motion on a large class of such surfaces is ergodic and even strongly chaotic [2]. Finite-area non-compact surfaces with points infinitely far away provide well known examples of chaotic scattering, with or without the inclusion of a constant magnetic field [3–7]. Moreover, such surfaces capture some of the features of multiply connected mesoscopic systems too. After threading the handles with Aharonov–Bohm fluxes in the presence of a constant magnetic field, the persistent currents, the adiabatic charge transport and the Hall conductances can be calculated exactly [8]. In this paper, combining the aforementioned ideas, we present a model of chaotic Aharonov–Bohm scattering. This problem was originally posed in a paper by Gutzwiller [9], studying a model similar to ours without attempting to solve it. Indeed, in the physics literature to the best of the author's knowledge, no such model was explicitly solved. Models of chaotic scattering included only single or multichannel systems, with or without the inclusion of a constant magnetic field. In those calculations the fluctuating part of the scattering matrix expressed in terms of the Riemann famous zeta-function always turned out to be independent of the magnetic field. Moreover, it is by now a common belief that for such systems we does not always have to come across the Riemann zeta-function. In spite of this, the explicit demonstration of the occurrence of other number-theoretic functions for the fluctuating part *also depending on the fluxes* is still missing. The aim of this paper is to fill in this gap, and *for a special choice* of fluxes present an exact solution to Gutzwiller's problem.

0305-4470/00/224129+13$30.00

This paper is organized as follows. In section 2 using the notation of Gutzwiller's paper [9] we present our leaky box. In section 3 our special choice of fluxes is introduced, which is amenable for an explicit calculation of the scattering matrix. In section 4 well known results concerning scattering theory on the relevant Riemann surfaces are summarized. The scattering matrices are calculated in section 5. The next section is devoted to a discussion of scattering resonances, and their relationship with the non-trivial zeros of some Dirichlet L-function. Section 7 gives some comments and the conclusions.

2. A topological model of the Aharonov–Bohm set-up

First we present our leaky box $\Sigma_{1,2}$, modelling the Aharonov–Bohm set-up. As shown in figure 1 this is a torus with one handle, and two leaks or cusps, i.e. points infinitely far away, describing the two scattering channels. Since the box is empty we can introduce two Aharonov–Bohm fluxes Φ_1 and Φ_2 associated with the two non-contractible loops of the handle. A charged particle can enter and leave any of the leaks. We are interested in calculating the scattering matrix as a function of the scattering energy and the fluxes.

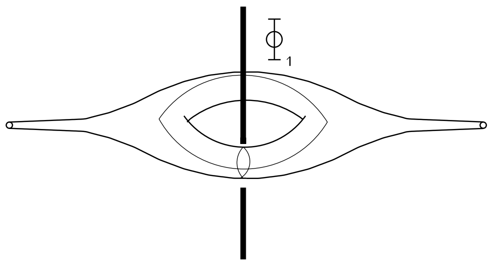

Figure 1. A torus, with one handle, and two leaks (cusps). The loops giving rise to the two fluxes are also depicted. Φ_1 is the flux threading the handle.

In order to construct this surface we cut a fundamental domain out of the Poincaré upper half-plane $\mathcal{H} \equiv \{z = x + \mathrm{i}y \in \mathbb{C} | y > 0\}$, with metric $\mathrm{d}s^2 = y^{-2}((\mathrm{d}x)^2 + (\mathrm{d}y)^2)$ of Gaussian curvature $K = -1$, and 'glue' the boundary points with the help of a *suitable* subgroup Γ of $PSL(2, \mathbb{Z})$ having a finite number of generators [5, 6]. Here $PSL(2, \mathbb{Z})$ is the group of fractional linear transformations $\gamma = (ab|\ cd) \in SL(2, \mathbb{Z})$, $\gamma z = \frac{az+b}{cz+d}$ with the properties $a, b, c, d, \in \mathbb{Z}$ and $ad - bc = 1$. In this way we can regard our surface $\Sigma_{1,2}$ as the right coset $\Gamma \backslash \mathcal{H}$.

How do we find a suitable Γ for constructing $\Sigma_{1,2}$? For its generators we search among the unimodular 2×2 matrices of *integer* entries, i.e. elements of $SL(2, \mathbb{Z})$. Two generators T and S are associated with the leaks, and two generators V and W with the non-contractible loops of figure 1. Moreover, these generators have to satisfy the constraint $S = V^{-1}WT^{-1}VW^{-1}$ [9]. A set of four matrices satisfying such constraints was found in [9]. However, as stressed by Gutzwiller himself, in this representation the calculation of the Aharonov–Bohm scattering matrix cannot be carried through thanks to an as yet unsolved mathematical problem. Here we use a different representation of the leaky box. This representation enables us to solve Gutzwiller's problem *for a special choice of fluxes*.

The basic idea is to search for the generators S, T, V and W among the matrices belonging to the Hecke-congruence subgroup $\Gamma_0(p)$ of $SL(2, \mathbb{Z})$, where p is an arbitrary prime number. These groups are defined as

$$\Gamma_0(p) \equiv \left\{ \gamma \in SL(2, \mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\ p) \right\}. \qquad (1)$$

It is known, that for such groups the surfaces $\Gamma_0(p)\backslash\mathcal{H}$ have exactly *two* leaks. Moreover, for the number of handles g (the genus) we have the formula (see proposition 1.40) of [10] $g = 1 + \frac{1}{12}\mu - \frac{1}{4}\nu_2 - \frac{1}{3}\nu_3 - \frac{1}{2}\nu_\infty$, where $\mu = p + 1$ is the index of $\Gamma_0(p)$ in $SL(2,\mathbb{Z})$ and $\nu_\infty = 2$ is the number of leaks. ν_2 and ν_3 are the number of elliptic points of order two and three. Their presence renders our surface *not* a smooth manifold. For p prime we have $\nu_2 = 1 + \left(-\frac{1}{p}\right)$ and $\nu_3 = 1 + \left(-\frac{3}{p}\right)$, where $\left(\frac{-}{p}\right)$ is the quadratic residue symbol known from number theory. In order to have $\nu_2 = \nu_3 = 0$ we have to satisfy the system of congruences $p \equiv 3 \bmod 4$ and $p \equiv 2 \bmod 3$. Since $(4, 3) = 1$ according to the Chinese remainder theorem we have only one solution modulo 12. The smallest prime satisfying this system is $p = 11$. Using this we obtain for the number of handles the value $g = 1$. Hence it should be possible to construct our leaky box $\Sigma_{1,2}$ by using the group $\Gamma_0(11)$. In order to find the generators and the fundamental domain we have to repeat the steps of [9] with matrices belonging to $\Gamma_0(11)$. For them with some luck we have found the following representation:

$$T = -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} -1 & 0 \\ 11 & -1 \end{pmatrix} \qquad V = \begin{pmatrix} 2 & 1 \\ 11 & 6 \end{pmatrix} \qquad W = \begin{pmatrix} 3 & 2 \\ 22 & 15 \end{pmatrix}. \quad (2)$$

A fundamental domain for this choice can be seen in figure 2. The two leaks are at the points i∞ and 0. The arrows indicate that the identifications have to be made by the corresponding transformations when building our box. Indeed, it is easy to check that $(-\frac{3}{4}, -\frac{2}{3}) \mapsto (\frac{1}{6}, 0)$ by W, $(-\frac{2}{3}, 0) \mapsto (\frac{1}{4}, \frac{1}{6})$ by V, and $-\frac{3}{4} \mapsto \frac{1}{4}$ by T as it has to be. Moreover, the constraint $S = V^{-1}WT^{-1}VW^{-1}$ [9] is also satisfied. It is instructive to compare the (2) generators with those of Gutzwiller [9]. His fundamental domain was symmetric (unlike ours).

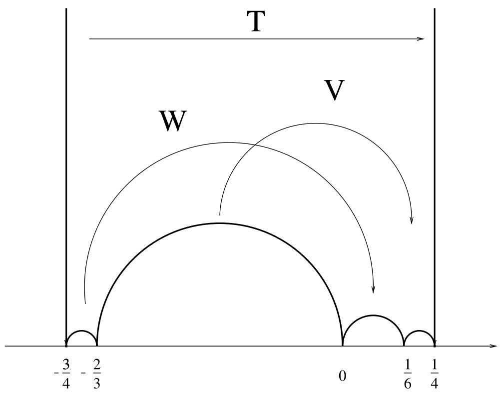

Figure 2. A fundamental domain for the group $\Gamma_0(11)$.

3. A special choice of fluxes

In order to introduce a set of values for the pair (Φ_1, Φ_2) of fluxes we have to find a mathematical way of assigning phase factors $e^{ie\Phi_1/\hbar c}$ and $e^{ie\Phi_2/\hbar c}$, to the generators V and W, respectively. Then this correspondence will fix the set of Aharonov–Bohm fluxes picked up by the wavefunction when winding around the loops of figure 1. Moreover, with the generators T and S we associate no fluxes modulo the flux unit. This means that when winding round our leaks no non-trivial phase factor is picked up. As pointed out in [8] threading the leaks by non-trivial fluxes renders the corresponding scattering channel closed unless the quantization condition $\Phi \equiv n\Phi_0$ with $n \in \mathbb{Z}$ holds, where $\Phi_0 = hc/e$ is the flux quantum.

The correspondence we are looking for, is a $U(1)$ representation $\mathcal{D}$ for the group $\Gamma_0(11)$. Such representations for the groups $\Gamma_0(p)$ are easily constructed via the use of Dirichlet characters [11]. (In fact, this was the main reason for choosing these groups as our candidates for model building.) Indeed, for a Dirichlet character to modulus p we have the correspondence [12]

$$\mathcal{D}(\gamma) \equiv \chi(d) \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p). \tag{3}$$

χ is a complex-valued periodic completely multiplicative arithmetical function with period p, and with the property $\chi(n) = 0$ if n is divisible by p. It is well known that there are $p - 1$ distinct Dirichlet characters. Hence in our case there are 10. An explicit formula for such characters is

$$\chi_h(n) = \exp\left[2\pi \mathrm{i}\frac{hb(n)}{10}\right] \qquad h = 1, \ldots, 10 \tag{4}$$

where h labels the distinct characters and $b(n)$ is the index of the integer n with respect to a primitive root [11]. Choosing the smallest primitive root, which is 2 for $p = 11$, we obtain the following set of values $(b(1), b(2), b(3), b(4), b(5), b(6), b(7), b(8), b(9), b(10)) = (0, 1, 8, 2, 4, 9, 7, 3, 6, 5)$. Hence for the generators of equation (2) we have to use the values $b(6) = 9$ for V, and the value $b(15) = b(4) = 2$ for W. For different choices of h we find a different set of values for the fluxes. These values are $(\Phi_1, \Phi_2) = (\frac{9}{10}h, \frac{2}{10}h)\Phi_0$, where $h = 1, \ldots, 10$. Moreover, since the cusps are not plugged, they are *not* threaded by Aharonov–Bohm fluxes. According to equations (2) and (4) this means that we must have $hb(-1) \equiv hb(10) \equiv hb(1) \bmod 10$. Since $b(1) = 0$ and $b(10) = 5$ the allowed values for h which are capable of describing a scattering situation are $h = 0, 2, 4, 6, 8$. These characters are precisely the *even* Dirichlet characters, i.e. those having the property $\chi(-1) = \chi(1) = 1$. The $h = 0$ choice describes the situation when no fluxes are present. For $h = 2, 4, 6, 8$ the set of values for the fluxes we are interested in can be described as

$$(\Phi_1, \Phi_2) = \tfrac{1}{5}(n_1, n_2)\Phi_0 \qquad n_1 = 4, 3, 2, 1 \qquad 2n_1 + n_2 \equiv 0 \mod 5. \tag{5}$$

Hence the representation $\mathcal{D}$ of $\Gamma_0(p)$ is given by fixing the representation of the generators by the formulae

$$\mathcal{D}(V) = \mathrm{e}^{2\pi \mathrm{i}\Phi_1/\Phi_0} \qquad \mathcal{D}(W) = \mathrm{e}^{2\pi \mathrm{i}\Phi_2/\Phi_0} \qquad \mathcal{D}(T) = \mathcal{D}(S) = 1 \tag{6}$$

where the allowed set of values for the fluxes is given by (5). It is important to realize that according to (5) the fluxes Φ_1 and Φ_2 are not independent. Hence it is enough to fix merely one of the fluxes, e.g. the flux Φ_1 of figure 1 in order to fix the physical situation.

4. Chaotic Aharonov–Bohm scattering

Having constructed our box, and fixed a special subset of fluxes through the non-contractible loops corresponding to V and W, we now turn to the quantum dynamics of a charged particle in the box. This dynamics arises from the quantization of the *classically* chaotic free (geodesic) motion of a particle with charge e and mass m on our surface. It is described by the Hamiltonian

$H = -\frac{\hbar^2}{2m}(\Delta + \frac{1}{4})$, where $\Delta \equiv y^2(\partial_x^2 + \partial_y^2)$ is the Laplace operator associated with the Poincaré metric on $\mathcal{H}$ [5]. Introducing the wavenumber as $k^2 = 2mE/\hbar^2$ Schrödinger's equation for the scattering states can be written as

$$(\Delta + \tfrac{1}{4} + k^2)\Psi(z) = 0 \qquad z \in \mathcal{H}. \tag{7}$$

Note that the classical dynamics is *not* affected by the fluxes. Hence classical trajectories are not changed. However, as we have learnt from the Aharonov–Bohm effect, quantization means more than assigning a trajectory for a particle to run in. This extra information is contained in the special form of the boundary condition we enforce on the wavefunction,

$$\Psi(\gamma z) = \mathcal{D}(\gamma)\Psi(z) \qquad \gamma \in \Gamma_0(11) \qquad z \in \mathcal{H}. \tag{8}$$

Every element $\gamma \in \Gamma_0(11)$ is a word built from the letters T, S, V, W. These letters are associated with homotopy classes of loops and a prescribed set of fluxes. Hence the boundary condition above describes the wavefunction picking up an Aharonov–Bohm phase factor when going round the loop corresponding to γ.

Our task is now to find the scattering solutions of (7) satisfying the boundary condition (8). Such scattering states are described by the Eisenstein series. For physicists a good place to read about them is [6]. In the following we give an account of them for the groups $\Gamma_0(p)$, where p is a prime of the form $p = 11 + 12r$, $r = 0, 1 \ldots$. For each such value we have a smooth surface with genus $g = (p + 1)/12$ and *two* cusps which can be chosen to be those at $\mathrm{i}\infty$ and 0. Since $(11, 12) = 1$, according to Dirichlet's famous theorem [13] we have an infinite number of primes of the form $p = 11 + 12r$, and hence an infinite number of such *smooth* surfaces with *two* cusps. Their genus is $g = r + 1$, $r = 0, 1, 3, 4, 5, 6, 8, 10, \ldots$. The case we explicitly worked out in this paper is that with $r = 0$. Let us label the cusps by α, which can take the values ∞ and 0. Moreover, let us denote the infinite cyclic groups consisting of elements of the form T^n and S^n, with $n \in \mathbb{Z}$ by Γ_∞ and Γ_0. Elements of Γ_α are fixing the cusps at z_α. The cusp at $z = \mathrm{i}\infty$ is called the *standard cusp*. The cyclic group fixing it generated by the element $-T$ will be denoted by $\mathcal{B}$. It is clear that $\Gamma_\infty = \mathcal{B}$. One can choose an element $\sigma_\alpha \in SL(2, \mathbb{R})$ such that $\sigma_\alpha(\mathrm{i}\infty) = z_\alpha$. Clearly, $\sigma_\infty = (1, 0|0, 1)$, and for the other cusp we choose $\sigma_0 = (0, -1/\sqrt{p}|\sqrt{p}, 0)$, in our case $p = 11$. These matrices transform our cusp to the standard one, moreover, $\sigma_\alpha^{-1}\Gamma_\alpha\sigma_\alpha = \mathcal{B}$. It is straightforward to check that the wavefunctions $\varphi_\pm(z) \equiv \mathrm{Im}(z)^{\frac{1}{2}\pm\mathrm{i}k} = y^{\frac{1}{2}\pm\mathrm{i}k}$ satisfy the Schrödinger equation (7), and represent incoming and outcoming plane waves for the standard cusp. In order to also satisfy the (8) boundary condition we construct the Eisenstein series for the cusp α as

$$E_\alpha(z, s, \mathcal{D}) = \sum_{\gamma \in \Gamma_\alpha \backslash \Gamma} \overline{\mathcal{D}}(\gamma)\, \mathrm{Im}(\sigma_\alpha^{-1}\gamma z)^s \qquad \alpha = \infty, 0 \tag{9}$$

where the overbar denotes complex conjugation and $\mathcal{D}$ is taken from the $U(1)$ representations defined by (5) and (6). The series is only defined for $\mathrm{Re}\, s > 1$, but an analytic continuation exists [12]. Moreover, $E_\alpha(z, s, \mathcal{D})$ satisfies the boundary condition (8), and the equation $(\Delta + s(1 - s))E_\alpha(z, s, \mathcal{D}) = 0$, for each $\alpha \in \{\infty, 0\}$. Hence with the choice $s = \frac{1}{2} + \mathrm{i}k$ it satisfies the Schrödinger equation (7) too.

In order to calculate the scattering matrix $S_{\alpha\beta}(k, \mathcal{D})$ $\alpha, \beta \in \{\infty, 0\}$, we need to know how E_α behaves near the cusp β. One can derive a Fourier expansion of E_α at the cusp β, the result is [12]

$$E_\alpha(\sigma_\beta z, s, \mathcal{D}) = \delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s, \mathcal{D}) y^{1-s} + \sum_{n \neq 0} \varphi_{\alpha\beta}(n, s, \mathcal{D}) W_s(nz). \tag{10}$$

Here $\varphi_{\alpha\beta}(n, s, \mathcal{D})$ $n = 0, 1, \ldots$ are the Fourier coefficients, and

$$W_s(z) = 2|y|^{1/2} K_{s-1/2}(2\pi|y|)\,\mathrm{e}^{2\pi \mathrm{i}x} \tag{11}$$

is Whittaker's function. A general formula for these coefficients has been determined by number theorists. Here we only need $\varphi_{\alpha\beta}(s, \mathcal{D}) \equiv \varphi_{\alpha\beta}(0, s, \mathcal{D})$, i.e. the zeroth one, which is of the form [12]

$$\varphi_{\alpha\beta}(s, \mathcal{D}) = \pi^{1/2} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \sum_{c>0} c^{-2s} Q_{\alpha\beta}(c, \mathcal{D}) \tag{12}$$

where

$$Q_{\alpha\beta}(c, \mathcal{D}) = \sum_{\gamma=(**|cd)\in\mathcal{B}\backslash\sigma_\alpha^{-1}\Gamma\sigma_\beta/\mathcal{B}} \overline{\mathcal{D}}\left(\sigma_\alpha \begin{pmatrix} * & * \\ c & d \end{pmatrix} \sigma_\beta^{-1}\right). \tag{13}$$

This can be justified by the asymptotic behaviour of Whittaker's function $W_s(z) \sim \mathrm{e}^{-2\pi y}$ as $y \to \infty$. Using this the terms containing the non-zero Fourier coefficients in equation (10) are dying out exponentially when we approach any of our cusps, hence only the term $\delta_{\alpha\beta} y^s + \varphi_{\alpha\beta}(s, \mathcal{D}) y^{1-s}$ survives. Since after analytical continuation to $s = \frac{1}{2} + \mathrm{i}k$, y^s and y^{1-s} correspond to the incoming and outgoing plane waves, we are left with the correct asymptotic behaviour for scattering states. Moreover, from this it is clear that $\varphi_{\alpha\beta}\left(\frac{1}{2} + \mathrm{i}k\right)$ has to be proportional to the scattering matrix $\boldsymbol{S}_{\alpha\beta}(k, \mathcal{D})$. Indeed, it is convenient to define $\boldsymbol{S}_{\alpha\beta}(\lambda, k, \mathcal{D}) = \lambda^{-2\mathrm{i}k} \varphi_{\alpha\beta}\left(\frac{1}{2} + \mathrm{i}k, \mathcal{D}\right)$, where $0 < \lambda \in \mathbb{R}$ is arbitrary. For the physical meaning of λ see [2, 4, 5]. Here we only remark that a choice of λ tells us where the particle is registered after being scattered.

5. Calculation of the scattering matrix

Now we calculate $\boldsymbol{S}_{\alpha\beta}$ for our leaky box. Since the calculation is just the same for *all* primes p, we substitute $p = 11$ in the end. For our calculations we use the explicit form (12) of $\varphi_{\alpha\beta}(s, \mathcal{D})$.

First, we remark that a non-trivial element of the double coset in (13) is determined uniquely by the value of $c > 0$, and the value of d modulo c [12]. Let us denote by $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(p)$ the matrix $\sigma_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sigma_\beta^{-1}$, which is needed in (13). With this notation $Q_{\alpha\beta}(c, \mathcal{D}) = \sum_{0<d<c} \overline{\chi}(D)$, and $c > 0$. For the calculation of this sum, we have to clarify the relationship between d and D, etc for each pair of values for α and β. For this we simply have to calculate the matrix product $\sigma_\alpha^{-1} \gamma \sigma_\beta$ using the matrices $\sigma_\infty = (1, 0|0, 1)$, and $\sigma_0 = (0, -1/\sqrt{p}|\sqrt{p}, 0)$ with the labels α, and β from the set $\{0, \infty\}$.

Let us first consider the case of *non-trivial* fluxes. The case of trivial fluxes will be considered later. For the calculation of $Q_{\infty\infty}$ we have $d = D > 0$, and $c = C = np > 0$ with n a positive integer. Since χ is periodic with period p, the sum $Q_{\alpha\beta}(c, \mathcal{D}) = \sum_{0<d<c} \overline{\chi}(D)$ is simply n times the sum taken from $D = 1$ to $p-1$ ($\chi(p) = 0$). The numbers $1, \ldots, p-1 \bmod p$ in this sum form an Abelian group under multiplication, and $\bar{\chi}$ is a character of this group. Due to theorem 6.10 of [11] this sum is zero unless $\bar{\chi}$ is trivial. Hence $Q_{\infty\infty} = 0$ yielding the $\boldsymbol{S}$-matrix element $\boldsymbol{S}_{\infty\infty} = 0$. For the calculation of the Q_{00} sum, we note that now $d = A > 0$, and $c = -Bp > 0$. The condition for the determinant of γ to be one gives $AD \equiv 1 \bmod p$. It is then easy to see, using the periodicity of χ, that the sum can again be taken from $A = 1$ to $p - 1$, i.e. a reduced residue system mod p. Indeed, as A is running through the values of the reduced residue system the solutions D of the congruence $AD \equiv 1 \bmod p$ will do the same.

Due to the aforementioned property of Dirichlet characters we again get $S_{00} = 0$. Hence we obtained the important result that for the (5) choice of fluxes we have *no reflection*, i.e. a particle entering one of the cusps must come out of the other.

For the calculation of $Q_{\infty 0}(c, \mathcal{D}) = \sum_{0<d<c} \overline{\chi}(D)$, we have $c = D\sqrt{p} > 0$ and $d = -C/\sqrt{p} > 0$. Since $-C = np$ with n a positive integer $Q_{\infty 0}(c, \mathcal{D}) = \sum_{0<n<D} \overline{\chi}(D) = \overline{\chi}(D)\phi(D)$, where $\phi(D)$ is Euler's function, i.e. the number of integers less then D having the property $(n, D) = 1$. ($AD - BC = AD + Bnp = 1, A, B \in \mathbb{Z}$ implies $(n, D) = 1$.) For $Q_{0\infty}$ we have $c = -A\sqrt{p} > 0$ and $d = -B\sqrt{p} > 0$. Now let $n = -B > 0$ then $Q_{0\infty}(c, \mathcal{D}) = \sum_{0<n<-A} \overline{\chi}(D)$. Now using the relation $AD \equiv 1 \bmod p$ we have $\overline{\chi}(A)\overline{\chi}(D) = 1$. Hence $\overline{\chi}(D) = \chi(A) = \chi(-A)$, where in the last step we used the fact that χ is even. Using this we find $Q_{0\infty}(c, \mathcal{D}) = \chi(-A)\phi(-A)$.

For the calculation of the sum over $c > 0$ in (12) we recall that $c = \sqrt{p}D$ for the $\infty 0$ case and $c = -\sqrt{p}A$ for the 0∞ case. These sums give $p^{-s}\sum_{D>0} D^{-2s}\overline{\chi}(D)\phi(D)$, and $p^{-s}\sum_{-A>0}(-A)^{-2s}\chi(-A)\phi(-A)$, respectively. (Due to the determinant condition in these sums we should also have to include the constraints $(A, p) = 1$ and $(D, p) = 1$, but since $\chi(D) = \chi(A) = 0$ for $p|A$ and $p|D$, the terms *not* satisfying these constraints are automatically killed by the Dirichlet character.) Since $\chi(n)\phi(n)$ is a multiplicative function we can have an Euler product representation over all primes for the aforementioned two sums using theorem 11.7 of [11],

$$\sum_{n>0} n^{-2s}\chi(n)\phi(n) = \prod_q \left(1 + \frac{\chi(q)\phi(q)}{q^{2s}} + \frac{\chi(q^2)\phi(q^2)}{q^{4s}} + \cdots\right) \tag{14}$$

where the infinite product is for *all* primes q. Since $\phi(q^n) = q^{n-1}(q-1)$ and $\chi(q^n) = \chi^n(q)$ we have

$$\sum_{n>0} n^{-2s}\chi(n)\phi(n) = \prod_q \left(1 + (q-1)\frac{\chi(q)}{q^{2s}}\left[1 + \left(\frac{q\chi(q)}{q^{2s}}\right) + \left(\frac{q\chi(q)}{q^{2s}}\right)^2 + \cdots\right]\right). \tag{15}$$

Since $|\chi(q)| = 1$ (or 0 for $q = p$) and $|q^{1-2s}| < 1$ for $\mathrm{Re}\, s > 1$, as the sum in $[\cdots]$ is a geometric sum it is convergent and gives $(1 - \chi(q)q^{1-2s})^{-1}$. Using these results we finally obtain

$$\sum_{n>0} n^{-2s}\chi(n)\phi(n) = \prod_q \frac{1 - \chi(q)/q^{2s}}{1 - \chi(q)/q^{2s-1}} = \frac{L(2s-1, \chi)}{L(2s, \chi)} \tag{16}$$

where we have introduced Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_q \left(1 - \frac{\chi(q)}{q^{2s}}\right)^{-1} \tag{17}$$

moreover, we immediately wrote it in the form of an Euler product. The infinite products are absolutely convergent provided the sums are. It is the case for $\mathrm{Re}\, s > 1$. Since we need the value $s = \frac{1}{2} + \mathrm{i}k$ we will need again an analytic continuation. The analytic continuation exists, and can be established by the same techniques as were applied to the Riemann zeta-function [11]. Applying the same line of reasoning for the sum over D as well, we obtain the final result for the matrix $\varphi_{\alpha\beta}(s, \chi)$ as

$$\varphi_{\alpha\beta}(s, \chi) = \sqrt{\pi}\, p^{-s}\frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} \begin{pmatrix} 0 & \dfrac{L(2s-1, \overline{\chi})}{L(2s, \overline{\chi})} \\ \dfrac{L(2s-1, \chi)}{L(2s, \chi)} & 0 \end{pmatrix}. \tag{18}$$

Note that in these formulae the dependence on the fluxes manifests itself via the presence of Dirichlet characters. For the $p = 11$ case, the *four* different choices of even characters, correspond to the choices of fluxes (5) threading the non-contractible loops of figure 1.

Now we turn to the case of trivial fluxes. Repeating the same steps as was done previously it is not hard to see that for the sum over $c > 0$ in (12) we get $\sum_{n>0,p|n} n^{-2s}\phi(n)$ for the $\infty\infty$ and 00 cases, and $p^{-s}\sum_{n>0,(p,n)=1} n^{-2s}\phi(n)$ for the $\infty 0$ and 0∞ ones. Since the cases $p|n$ and $(p, n) = 1$ are complementary to each other it is enough to calculate merely one of the sums. Using again the Euler product representation (take a look at equation (16) with χ now being the trivial character corresponding to the $h = 0$ choice in (4)) we obtain

$$\sum_{n>0,(p,n)=1} n^{-2s}\phi(n) = \prod_{q,q\neq p}\frac{q^{2s}-1}{q^{2s}-q} = \frac{p^{2s}-p}{p^{2s}-1}\prod_{q}\frac{q^{2s}-1}{q^{2s}-q} = \frac{p^{2s}-p}{p^{2s}-1}\frac{\zeta(2s-1)}{\zeta(2s)} \tag{19}$$

where $\zeta(s)$ is Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_q\left(1-q^{-2s}\right)^{-1} \qquad \operatorname{Re} s > 1 \tag{20}$$

where again we wrote the Euler product representation.

After calculating the sum $\sum_{n>0,p|n} n^{-2s}\phi(n)$ after using $1 - (p^{2s} - p)/(p^{2s} - 1) = (p - 1)/(p^{2s} - 1)$ we obtain for the matrix $\varphi_{\alpha\beta}(s, 1)$ the final result

$$\varphi_{\alpha\beta}(s) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{1}{p^{2s}-1}\begin{pmatrix} p-1 & p^s-p^{1-s} \\ p^s-p^{1-s} & p-1\end{pmatrix}. \tag{21}$$

As we have already mentioned the analytical continuation of these matrices exists, so we can put in the value $s = \frac{1}{2} + \mathrm{i}k$ into these formulae. Moreover, we can use the functional equation of the Dirichet L-function and the Riemann zeta-function to express these results in a more instructive form. For an *even* Dirichlet character we have [13]

$$\frac{\pi^{1/2}}{\tau(\overline{\chi})}\pi^{-s/2}p^{s/2}\Gamma(s/2)L(s,\overline{\chi}) = \pi^{-\frac{1}{2}(1-s)}p^{\frac{1}{2}(1-s)}\Gamma((1-s)/2)L(1-s,\chi) \tag{22}$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{n=1}^{p}\chi(n)\,\mathrm{e}^{2\pi\mathrm{i}n/p}. \tag{23}$$

It is well known that for $h \neq 0$, $|\tau(\chi)| = p^{1/2}$. Moreover, our characters of (4) for $p = 11$ with $h = 2, 4, 6, 8$ are *even*, hence $\overline{\tau(\chi)} = \tau(\overline{\chi})$. Collecting these results we obtain the final form for our Aharonov–Bohm scattering matrix

$$\begin{aligned} \boldsymbol{S}_{\alpha\beta}(s,\lambda,\chi) &= \left(\frac{\lambda}{\pi}\right)^{-2\mathrm{i}k} p^{-3\mathrm{i}k}\frac{\Gamma\left(\frac{1}{2}-\mathrm{i}k\right)}{\Gamma\left(\frac{1}{2}+\mathrm{i}k\right)} \\ &\quad\times\begin{pmatrix} 0 & \dfrac{1}{\sqrt{p}}\overline{\tau(\chi)}\dfrac{L(1-2\mathrm{i}k,\chi)}{L(1+2\mathrm{i}k,\overline{\chi})} \\ \dfrac{1}{\sqrt{p}}\tau(\chi)\dfrac{L(1-2\mathrm{i}k,\overline{\chi})}{L(1+2\mathrm{i}k,\chi)} & 0\end{pmatrix}. \end{aligned} \tag{24}$$

Note that the non-zero (off-diagonal) matrix elements are phase factors, and the determinant of the scattering matrix is independent of the Gauss sum. Moreover, the $\boldsymbol{S}$-matrix is unitary as it has to be.

Similarly, using the functional equation for the Riemann zeta-function, the scattering matrix of trivial fluxes can be cast into the form

$$\boldsymbol{S}_{\alpha\beta}(k,\lambda) = \left(\frac{\lambda}{\pi}\right)^{-2ik} \frac{\Gamma\left(\frac{1}{2}-\mathrm{i}k\right)}{\Gamma\left(\frac{1}{2}+\mathrm{i}k\right)} \frac{\zeta(1-2\mathrm{i}k)}{\zeta(1+2\mathrm{i}k)} \mathcal{R}_{\alpha\beta}(p,k) \tag{25}$$

where

$$\mathcal{R}(p,k) = M_p^{-1}\left(\tfrac{1}{2}+\mathrm{i}k\right)M_p\left(\tfrac{1}{2}-\mathrm{i}k\right) \qquad M_p(s) = \begin{pmatrix} 1 & p^s \\ p^s & 1 \end{pmatrix}. \tag{26}$$

The case we are interested in is of course the one with $p = 11$. For this case we have explicitly worked out the correspondence between Dirichlet characters and Aharonov–Bohm fluxes. However, by finding a set of generators for the groups $\Gamma_0(p)$ for other primes of the form $11+12n$, this correspondence between fluxes and characters can, in principle, be found. Then the scattering matrix (24) is describing a (two-channel) Aharonov–Bohm scattering situation on a $g = n + 1$ leaky box. Of course in this case finding the correspondence between the $2g$ Aharonov–Bohm fluxes and the $p - 1$ Dirichlet characters is not a trivial task.

We close this section with some comments. The scattering matrix (25) is well known to number theorists. It was calculated first using more comprehensive methods than ours using Hejhal in [14, 15]. The less familiar scattering matrix (24) appeared first (without a proof) in the book of Iwaniec [12] and it was taken from unpublished notes of Pitt. The proof here (which is elementary) was reproduced by the author based on the scattered results found in [12]. As far as we know, hitherto used merely for number-theoretic considerations, neither of these scattering matrices have yet made their debut in the physics literature.

6. Resonances

Now we would like the clarify the physical meaning of our scattering matrix (24) with $p = 11$. First, we would like to see how the Aharonov–Bohm fluxes (5) enter the $\boldsymbol{S}$-matrix. From (24) we see that the fluxes enter via the presence of Dirichlet characters. The dependence on the Dirichlet characters is due to the k-independent Gauss sum and the k-dependent L-function. Let us consider the L-function $L(s,\chi)$, $s = \frac{1}{2} + \mathrm{i}k$. Looking at the definition (17) after decomposing n according to residue classes mod 11, i.e. having $n = 11m + r$ where $m = 0, 1, 2, \ldots$ and $1 \leqslant r \leqslant 11$, and using that the period of χ is 11, i.e. $\chi(11m + r) = \chi(r)$, we obtain

$$L(s,\chi) = 11^{-s}\sum_{r=1}^{11}\chi(r)\sum_{m=0}^{\infty}\frac{1}{(m+r/11)^s} = 11^{-s}\sum_{r=1}^{10}\chi(r)\zeta(s,r/11) \tag{27}$$

where we have used that $\chi(11) = 0$, and introduced the Hurwitz zeta-function

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \qquad \operatorname{Re} s > 0. \tag{28}$$

Here a is a fixed real number such that $0 < a \leqslant 1$. As we expect the analytic continuation of $\zeta(s,\chi)$ exists [11]. The Hurwitz zeta-function provides a good example of deformation in analytic number theory. For $a = 1$ we obtain the Riemann zeta-function, which according to (25) is related to the case when no fluxes are present. When Aharonov–Bohm fluxes are also present, according to (27) we have to use a linear combination of different deformations of the Riemann zeta-function. Since the fluxes are not independent, we can merely use Φ_1

to describe all values of the Dirichlet characters. We label the possible values of Φ_1 as $\Phi_1^{n_1}$ where $n_1 = 1, 2, 3, 4$ (see also (5)). With this notation, we can write

$$\begin{aligned}11^s L(s, \chi^{n_1}) = {} & \left(\zeta\left(s, \tfrac{1}{11}\right) + \zeta\left(s, \tfrac{10}{11}\right)\right) + \left(\zeta\left(s, \tfrac{5}{11}\right) + \zeta\left(s, \tfrac{6}{11}\right)\right) \exp\left[2\pi\mathrm{i}\frac{\Phi_1^{n_1}}{\Phi_0}1\right] \\ & + \left(\zeta\left(s, \tfrac{3}{11}\right) + \zeta\left(s, \tfrac{8}{11}\right)\right) \exp\left[2\pi\mathrm{i}\frac{\Phi_1^{n_1}}{\Phi_0}2\right] \\ & + \left(\zeta\left(s, \tfrac{4}{11}\right) + \zeta\left(s, \tfrac{7}{11}\right)\right) \exp\left[2\pi\mathrm{i}\frac{\Phi_1^{n_1}}{\Phi_0}3\right] \\ & + \left(\zeta\left(s, \tfrac{2}{11}\right) + \zeta\left(s, \tfrac{9}{11}\right)\right) \exp\left[2\pi\mathrm{i}\frac{\Phi_1^{n_1}}{\Phi_0}4\right]. \end{aligned} \tag{29}$$

From this formula it is clear that different choices for the flux yield different linear combinations of the pairs of deformed Riemann zeta-functions.

Now we would like to study the scattering resonances for the Aharonov–Bohm scattering. For this purpose we introduce the Wigner–Smith time delay [16] by the formula

$$\mathcal{T}(k, \lambda, \chi) = \frac{\mathrm{i}}{2k}\partial_k \log \mathrm{Det}\, \boldsymbol{S}(k, \lambda, \chi). \tag{30}$$

Note that our definition differs in sign from the usual one used, for example, in [6], due to a different identification of incoming and outgoing plane waves. The arbitrary parameter λ in this formula has the following meaning. Because the scattering is purely geometric, the role of interaction is not played by a potential. For usual scattering systems where the interaction is defined by a potential, the Wigner time delay is defined to be the difference between the time spent by the scattered particle within the region of the potential, and the time that it would have spent in the same region had it moved without the influence of the potential. Now interaction is just restriction of the motion from $\mathcal{H}$ to the fundamental domain of figure 2 via the use of the boundary condition (8). According to Gutzwiller [2] the value of λ can be regarded as a parameter regularizing the infinite length of the scattering trajectories. Its value gives a 'monitoring station' where the particle is registered after being scattered. In this picture the dependence of $\mathcal{T}$ on λ reflects the arbitrariness in the definition of the region of interaction needed for a definition of the time delay.

Now we would like to calculate $\mathcal{T}$ for the (24) scattering matrix. For this we note that $\mathrm{i}\partial_k \log\left((\frac{\pi}{\lambda})^{2\mathrm{i}k} p^{-3\mathrm{i}k}\right) = -2\log\frac{\pi}{\lambda} + 3\log p$, and

$$\mathrm{i}\partial_k \log\left(\frac{\Gamma\left(\frac{1}{2} - \mathrm{i}k\right)}{\Gamma\left(\frac{1}{2} + \mathrm{i}k\right)}\right) = 2\,\mathrm{Re}\,\psi\left(\tfrac{1}{2} + \mathrm{i}k\right) \tag{31}$$

where $\psi(z) \equiv \frac{\Gamma'}{\Gamma}(z)$ is the digamma function. Moreover, we also have

$$\mathrm{i}\partial_k \log\left(\frac{L(1 - 2\mathrm{i}k, \overline{\chi})}{L(1 + 2\mathrm{i}k, \chi)}\right) = 4\,\mathrm{Re}\left(\frac{L'(1 + 2\mathrm{i}k, \chi)}{L(1 + 2\mathrm{i}k, \chi)}\right). \tag{32}$$

Using formula (17) on p 83 of [13] for even characters, we obtain

$$\begin{aligned}4\,\mathrm{Re}\left(\frac{L'(1 + 2\mathrm{i}k, \chi)}{L(1 + 2\mathrm{i}k, \chi)}\right) = {} & -2\log\frac{p}{\pi} - 2\,\mathrm{Re}\,\psi\left(\tfrac{1}{2} + \mathrm{i}k\right) + 4\,\mathrm{Re}\,B(\chi) \\ & + 4\,\mathrm{Re}\sum_{\varrho} \frac{1 + 2\mathrm{i}k}{(1 + 2\mathrm{i}k - \varrho)\varrho} \end{aligned} \tag{33}$$

where ϱ denotes the non-trivial zeros of the $L(s, \chi)$-function, and [13]

$$\operatorname{Re} B(\chi) = -\sum_{\varrho} \operatorname{Re} \frac{1}{\varrho}. \tag{34}$$

Collecting everything we obtain

$$\mathrm{i}\partial_k \log \left(\left(\frac{\pi}{\lambda}\right)^{2\mathrm{i}k} p^{-3\mathrm{i}k} \frac{\Gamma\left(\frac{1}{2} - \mathrm{i}k\right)}{\Gamma\left(\frac{1}{2} + \mathrm{i}k\right)} \frac{L(1-2\mathrm{i}k, \overline{\chi})}{L(1+2\mathrm{i}k, \chi)} \right) = \log(p\lambda^2) + 4 \operatorname{Re} \sum_{\varrho} \frac{1}{1+2\mathrm{i}k - \varrho}. \tag{35}$$

For the calculation of $\mathcal{T}$ we have to use a similar formula to (35) with χ replaced by $\overline{\chi}$. Finally, we obtain

$$\mathcal{T}(k, \lambda, \chi) = \frac{2}{k} \left(\log 11\lambda^2 + \operatorname{Re} \sum_{\varrho} \frac{1}{1+2\mathrm{i}k - \varrho} + \operatorname{Re} \sum_{\overline{\varrho}} \frac{1}{1+2\mathrm{i}k - \overline{\varrho}} \right) \tag{36}$$

where $\overline{\varrho}$ are the non-trivial zeros of $L(1+2\mathrm{i}k, \overline{\chi})$. These are just the complex conjugates of ϱ. We note that for complex χ the zeros of $L(\sigma + \mathrm{i}t, \chi)$ are symmetric about the line $\sigma = \frac{1}{2}$, but *not* about the real axis. It is known that there is an infinity of non-trivial zeros all in the critical strip $0 \leqslant \sigma \leqslant 1$. There is also the conjecture (the generalized Riemannian hypothesis) that is attributed to Piltz in 1884 [13], that the function $L(s, \chi)$ has its zeros in the critical strip on the line $\sigma = \frac{1}{2}$. Denoting the non-trivial zeros as $\varrho = v_\varrho + 2\mathrm{i}k_\varrho$, and choosing $\lambda = 1/\sqrt{11}$ we obtain

$$\mathcal{T}(k, 1/\sqrt{11}, \chi) = \frac{1}{k} \left(\sum_{\varrho} \frac{\frac{1}{2}(1-v_\varrho)}{\frac{1}{4}(1-v_\varrho)^2 + (k-k_\varrho)^2} + \sum_{\varrho} \frac{\frac{1}{2}(1-v_\varrho)}{\frac{1}{4}(1-v_\varrho)^2 + (k+k_\varrho)^2} \right). \tag{37}$$

In the first sum the terms with $k_\varrho > 0$ (and in the second sum those with $k_\varrho < 0$) give a collection of Lorentzians with the positions of the resonances being at k_ϱ and their half-widths being $(1 - v_\varrho)/2$. Hence for sufficiently large $k > 0$ it is a good approximation to write

$$k\mathcal{T}(k, 1/\sqrt{11}, \chi) \sim \sum_{\varrho, k_\varrho > 0} \frac{\frac{1}{2}(1-v_\varrho)}{\frac{1}{4}(1-v_\varrho)^2 + (k-k_\varrho)^2} + \sum_{\varrho, k_\varrho < 0} \frac{\frac{1}{2}(1-v_\varrho)}{\frac{1}{4}(1-v_\varrho)^2 + (k+k_\varrho)^2}. \tag{38}$$

Hence in the neighbourhood of an isolated pole of the Aharonov–Bohm scattering matrix the dominant contribution to the time delay is a Lorentzian. The real parts of these complex poles are one-half of the imaginary parts of the zeros of a Dirichlet L-function, which is defined by the flux choices of equation (5). If the generalized Riemannian hypothesis is true the resonances have constant half widths $\Gamma/2 = \frac{1}{4}$. It would be nice to have a numerical investigation of the structure of these poles as was done in [17] for the case without Aharonov–Bohm fluxes. The scattering resonances in that case were associated with the zeros of the Riemann zeta-function. For an analysis in a similar spirit, the numerical values of the zeros of our L-functions would be needed. It would also be interesting to see numerically, how the position of the zeros (resonances) is changing as we go through the four different choices for our fluxes. Moreover, it is well known by now [18] that the presence of fluxes changes the eigenvalue statistics from GOE to GUE, due to the breaking of time-reversal invariance. It would also be nice to see what is the analogue of this transition in the statistics of resonances. For this purpose the general methods of [19] would be needed.

7. Conclusions

In this paper a topological version of Aharonov–Bohm scattering was presented. The Aharonov–Bohm set-up was modelled by a genus-one non-compact Riemann surface of constant negative curvature with two points (leaks) infinitely far away. A particle can enter on any one of the leaks and escape on any other. The resulting quantum mechanical scattering problem was a two-channel one. We have shown that for a special choice of the Aharonov–Bohm fluxes, the problem of calculating the scattering matrix as posed by Gutzwiller can be solved exactly. Our calculations show that for this choice of fluxes the scattering matrix has no diagonal entries, and hence we have no reflection. The fluctuating part of the off-diagonal entries (transmission rates) exhibit a remarkable energy dependence via a Dirichlet L-function. Unlike in situations discussed previously, now *the fluctuating part* shows a non-trivial dependence on the fluxes as well. We established a connection between the scattering resonances and the non-trivial zeros of the Dirichlet L-function. The distribution of these non-trivial zeros reflects the chaos of the underlying classical dynamics.

However, it is a non-trivial task to describe this correspondence more clearly. It is well known by now that without fluxes, we come across the Riemann zeta-function. The distribution of the famous Riemann zeros is now related to the scattering resonances of the flux-free case. Moreover, it is also known that we can relate the Riemann zeros to the error term in the prime number theorem. Loosely speaking the apparent irregularity observed in the distribution of primes is related to the chaotic nature of the scattering problem.

It is tempting to generalize this reasoning to the case when fluxes are also present. Dirichlet L-functions describe the distribution of primes in *arithmetic progressions* [13]. For our leaky box the primes in progressions are of the form $p \equiv a \bmod 11$, where a is a fixed integer. The prime number theorem for arithmetic progressions states that the function $\pi(x; 11, a)$ which is the number of primes up to x that are congruent to $a \bmod 11$, behaves asymptotically as $\frac{\mathrm{li}\, x}{\Phi(11)} = \frac{\mathrm{li}\, x}{10}$, where $\mathrm{li}\, x = \int_2^x \frac{\mathrm{d}t}{\log t}$. Number theorists are interested in the error term, i.e. how this asymptotic behaviour is achieved. It is known [13] that the a-independent part of the error term comes essentially from the quantity

$$\frac{1}{10} \sum_{\chi} \sum_{\varrho} |x^{\varrho}/\varrho| \tag{39}$$

where the sum is over *all Dirichlet characters modulo 11*. We have ten Dirichlet characters, however, only the five even ones describe a scattering situation. These zeros of the corresponding L-functions, related to the scattering resonances, clearly give a contribution to the error term. However, in (39) we also have to include the odd ones corresponding to a choice of fluxes rendering the scattering channels closed. Hence we may conclude that although the chaotic nature of the Aharonov–Bohm scattering is somehow related to the distribution of primes in arithmetical progressions, the meaning of this correspondence is by no means clear.

Acknowledgment

Financial support from Országos Tudományos Kutatási Alap (OTKA), grant nos T029813, T029884 and T032453 is gratefully acknowledged.

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